CONSTRUCTION OF GEOMETRY PROOFS: A REGRESSION ANALYSIS OF KNOWLEDGE STRANDS

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The development of proofs in geometry continues to be problematic for many students. Battista (2007) argued that we need to address this issue by examining the relative role of general processes and task-specific knowledge in the development of proofs by students. The present study is driven by the assumption that development of geometric proof constitutes a problem-solving activity the understanding of which requires the untangling of knowledge components that drive the process. Our literature review resulted in the identification of three knowledge components that are relevant to geometry proofs: Content Knowledge of Geometry, Metacognitive Skills (general processes) and Mathematical Reasoning. As expected, regression analysis indicated that Content Knowledge of Geometry to be the most important predictor of success in the construction of proofs. We also found that Metacognitive Skills and Mathematical Reasoning to be playing a significant role in the process of proof development. Taken together, the results suggest that solution of geometry proof problems are inherently complex and involve a robust interplay between students' general and task-specific knowledge components during the search in the problem space. Our work also contributes to the on-going debate about the role of general skills in mathematical problem solving.

INTRODUCTION

Many students are reluctant to solve problems in geometry that require them to develop formal proofs, and, when they do decide to tackle them, their performances have been shown to be unsatisfactory (Herbst, 2002). This continuing malaise with proofs with geometry was evident in a study reported by Martin, McCrone, Bower and Dindyal (2005) which showed that students experienced difficulty in constructing proofs and that further studies about how individual students understand and generate proofs is an important line of inquiry.

Harel and Sowder (2007) showed that some students develop and activate 'proof schemes' when they are required to draw logical inferences. The notion of scheme or schema suggests that students' knowledge is multidimensional and structured. Harel et al.'s analysis is an important starting point for research that explores the kind of knowledge, understandings and reasoning processes that could drive and govern the process of construction of proofs. We suggest that content-specific information and general processes constitute significant components of proof schemes, and the investigation of the relative impact of these three knowledge components is an important area of inquiry.

THEORETICAL CONSIDERATIONS

Domain Specific Knowledge and Metacognitive Skills

The role of domain-specific knowledge and metacognitive skills in learning and problem-solving performance has been the subject of considerable interest in recent years (Zohar & Peled, 2008). Within geometry, and specifically, geometry proofs, Hanna and Barbeau (2008) identified different strands of mathematical knowledge. Such knowledge could be viewed as domain-specific or general. Domain-specific knowledge consists of concepts, principles, conventions and principles that are unique to that domain. Metacognitive (general) skills, on the other hand, refer to a cluster of skills that indicate one's knowledge and control of one's own cognition.

Sweller (1989) argued that rich domain-specific knowledge plays a prominent role in mathematical understanding and problem solving in comparison to general processes while Lawson (1989) suggested that both these strands of knowledge interact and complement each other. In a recent study, Bertholda, Nückles and Renkl (2007) showed that more effective learning is evidenced when students are taught to apply both cognitive (domain-specific) and metacognitive prompts. Thus, there are competing views on the relative role of these two knowledge variants not only in the solution of geometry proof problems but mathematical problems in general.

Development of Geometry Proofs

The construction of geometric proofs can be seen as a problem-solving activity where students draw on a body of domain-specific or content knowledge. For example, Euclidean geometry consists of a coherent body of content knowledge that includes understanding the characteristics of objects such as point, angle, triangle and shapes as well as knowledge of use of axiomatic reasoning. In addition to this content-specific knowledge, students need to utilise a range of general skills during the course of their search for the proofs. A feature of non-algorithmic approach to geometry proof problem solving is the difficulty of finding a starting point or a method for approaching the problem (Healey & Hoyles, 1998; Riess, Kleime and Heinze, 2001). Cognitive processes such as planning have a role in directing the search for strategies (Schoenfeld, 1992). As a control process, metacognition orchestrates the solution process.

Reasoning skills play two key roles during the solution of geometry proof problems. Firstly, reasoning facilitates the construction of logical arguments. During the course of development of proofs, the solver develops a series of well-connected arguments that are based on the process of reasoning. Van Hiele's levels (1999) provide useful framework for analysis of reasoning patterns that are involved in geometry proofs. According to this model, at the level of Formal Deduction (van Hiele's Level 4), students draw on their formal deductive abilities to the solution process. Students working at this level have acquired knowledge of geometric concepts about basic plane figures, geometric relationships, and use them to understand proof situations. A student who reasons at Level 4 understands the notions of mathematical postulates and theorems and can write formal proofs of theorems (Senk, 1989). While there is little doubt about the role of reasoning in proof development, how specific these processes anchor the construction proofs in the domain of geometry is an area that has received less attention.

Purpose and Research Questions

Based upon the above review of literature, the principal research question addressed in this study was 'what is the relative contribution of knowledge of geometry concepts, metacognitive skills and mathematical reasoning skills in supporting the solution of geometry proof problems?' The study aimed at examining the predictive value of three independent variables (*Metacognitive Skills,* MS; *Geometry Content Knowledge,* GCK; *Mathematical Reasoning Skills*, MRS) on the dependent variable (*Proof-Type Geometry problem-solving*, PTG). PTG represents a measure of students' outcomes to solution of geometry proof problems.

Participants

Participants in this study were Year 11 students from Sri Lanka (n=166). Year 11 students in Sri Lanka study mathematics that was based on a common curriculum that includes that key strands of Number, Algebra and Geometry. The participating students were enrolled in four high schools within the metropolitan areas of Colombo and Kandy. Within each school, the students were enrolled in a single mathematics class.

Tests and measures

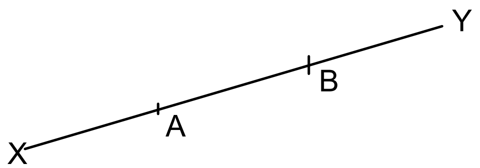
Tests were developed to assess students' PTG, MS, and GCK. Students' MRS scores were based on their performance in a common examinations conducted at the end of Grade 10 for all students in Sri Lanka. These examination included problems in geometry and algebra with a strong emphasis on working mathematically and reasoning skills.

Test development and scoring procedures

The items for the tests were selected from a pool of resources such as textbooks, examination papers and research papers. A number of items were modified for the purpose of the study. The items were reviewed by a group of six senior mathematics teachers (including one teacher from Australia) and a curriculum expert from the Sri Lankan Department of Education. Each test was piloted with a group of 15 students from a school that did not participate in the final data collection phase of the study. The duration for each of the tests was as follows: PTG – 80 minutes; MS – 80 minutes; GCK – 60 minutes.

Proof Type Geometry Problem Solving (PTG) Test

Five geometry problems of varying levels of difficulty were selected and modified for this test. All problems were well-structured proof problems. Our decisions to select or modify these problems were guided by a) richness and range of embedded geometric knowledge, b) opportunities to explore representations (MS) and c) multiple reasoning steps involved in the development of proofs. Figure 1 below is one of the problems that appeared in this test. The proof for this problem involves students to use a sequence of reasoning moves (MRS) to represent the problem in terms of appropriate equations. In so doing, students need to activate geometry content specific knowledge items such as equality and segments (GCK).



The line AB has been extended to either side so that AX = BY. Prove that AY = BX.

Figure 1

Metacognitive Skills (MS) Test

The MS test contained five non-routine problems. The written test was constructed so that students could present evidence related to four general processes of problem solving: (a) *analysis,* (b) *representation,* (c) *planning* and (d) *use of knowledge retrieval*. These four general processes emerged to be significant from our review of research on problem solving. Figure 2 is one of the problems that appeared in this test.

You are to organise a tea party for the class at the end of the year. How would you find out the food-item preferences of your classmates?

Figure 2

Scoring rubric for PTG and MS

The scoring procedures for PTG and MS were based on the same rubric. However, we focused on different parts of the rubric in order to generate the scores for PTG and MS. In developing the scoring rubric we aimed to include dimensions of levels and cognitive processes. The criteria for scoring were developed from geometry proof problem perspective. The scoring rubric included features of a two-dimensional matrix: process and level. The processes were: *analysis*, *representation*, *planning* and *use of knowledge retrieval*.

Geometry Content Knowledge (GCK) test

Geometry proof problem solving requires the activation of *Geometry Content Knowledge*. The GCK test was designed to measure students' acquisition of rules and declarative knowledge components that were required for the solution of five geometry proof problems of the PTG Test. This knowledge was broadly classified into 15 components because these were considered to be the content requirements for the solution of the five geometry proof problems that were the focus of the present study. Figure 3 shows one of the items that appeared in the GCK test.

Provide two pieces of information conveyed by the diagram.

ABC is a triangle

AB = AC

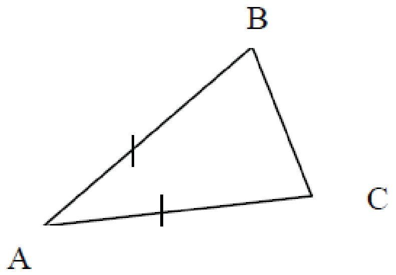


Figure 3: GCT test item

Student responses to Geometry Content Knowledge test was scored as: 1 - correct response;

0 - incorrect response.

Test administration procedures

A 2-hour session was conducted to practise answering the MS test. The problems used in the practice session were not similar to the test items in the final MS test because the aim was to practise written presentation of general problem solving. In order to avoid possible practice effects of the training session on performance in MS, the practice session was followed by the PTG test instead of MS. In order to avoid possible practice effects of the PTG on GCK, PTG was followed by the MS test instead of GCK. In order to avoid possible practice effects of the GCK on PTG, GCK was administered last. Students' regular classroom teachers administered all tests during their mathematics lessons.

RESULTS AND ANALYSIS

Correlation Analyses

Pearson correlation coefficients indicated that GCK is the independent variable that was most strongly related to geometry proof problem-solving performance. The other variables (MS and MRS) were also significantly correlated to PTG. This suggested that while GCK could be the major predictor of geometry proof problem-solving performance, students' general metacognitive skills (MS) and their reasoning processes (MRS) carried significant load in the regression model.

Regression analysis

To test our principal research question (What is the relative contribution of knowledge of geometry concepts, metacognitive skills and mathematical reasoning skills in supporting the solution of geometry proof problems?) a stepwise regression analyses was conducted. The Multiple Regression coefficient (R) is an important statistic in the regression analysis. It is the square root of the coefficient of determination or the correlation squared (R2), which is the total proportion of variation of the dependent variable explained by dependent variables. The results of the analysis are shown in Table 1.

Variables Entered	Variables Removed	R	R^2	Adjusted R^2	Std. Error of the Estimate
MRS MS GCK	-	.852	.726	.721	5.34

Dependent Variable: PTG; c Predictors: (Constant), GCK, MS, MRS.

Table 1: Model summary

According to the above model, 72.6% of variation in PTG was explained by the three variables: MRS, MS, and GCK. The adjusted R2 is an estimated value to use as the population estimator, as small samples tends to overfit. The difference between R2 and adjusted R2 was not large indicating the strength of the prediction. Analysis of variance showed that R2 was significant ($F3,_{162}$ =143.440, $p < 0.001$). Since the above R2 represents a collective effect, it could not be used to explain the variations in terms of the contribution from each independent variable. We, thus, computed regression coefficients for each of the independents. The unstandardized coefficients for the independent variable (all significant at $\alpha = 0.05$) allowed us to generate the following regression equation:

$$PTG = -6.399 + .939\, GCK + .314\, MS + .122\, MRS$$

DISCUSSION AND IMPLICATIONS

The aim of the study was to examine the relative contribution of students' *general processes* vs *task-specific knowledge* as they attempted to solve problems in the domain of geometry that involves the construction of proofs. Three knowledge components were hypothesized to be relevant for proof construction: *Geometry Content Knowledge, General Processes and Mathematical Reasoning Skills.*

The regression analysis suggested that all three knowledge components were involved in predicting students' success in solving a given set of geometry proof problems. In doing so, we found that *Geometry Content Knowledge* (GCK) was most influential in aiding students generate appropriate proofs. Both *General Processes* (MS) and *Mathematical Reasoning Skills* (MRS) made significant contributions to the activation and utilization of *Geometry Content Knowledge* by the students during the course of solution search. About 67% of the variance in the measures of performance in the development of proofs could be attributed to *Geometry Content Knowledge*. This suggests that improving *Geometry Content Knowledge* could have a significant positive impact on students' success in generating geometry proofs. The overwhelming impact of students' prior geometric knowledge evidenced in the present study is consistent with that reported by Senk (1985). In that study, which involved the participation of 2567 students from the United States, Senk obtained a value of 0.67 for Pearson correlation coefficient between geometry proof problem solving and students' content knowledge of geometry. This further demonstrates the importance of acquisition and use of a robust body of *Geometry Content Knowledge* schemas in geometry proof problem solving.

Studies have shown that domain knowledge is not a sufficient condition for the application of that knowledge during the course of solution search. For instance, Lawson and Chinnappan (in press) found that students who can be shown to have acquired the necessary geometry knowledge failed to utilise that knowledge during problem solving when it would have been appropriate to do so. Thus, it would seem that there are other knowledge components that work in concert with GCK.

Content knowledge related to geometry problem solving includes geometric concepts, knowledge about geometric relationships and visual representations of such relationships via appropriate diagrams. Reiss et al. (2001) argued that the above components need to be organized into meaningful schemas in order to foster accessibility. Geometry proof problems have a unique structure and the solution of such problems requires, firstly, an intuitive understanding of the given context and what the solution could be. This intuitive understanding, we suggest, involve students entertaining potential moves that would be productive.

Let us consider the second variable: *General Processes*. During the solution attempts, students, almost always, need to translate problem information into an appropriate diagram. This process of *representation* involves the translation of geometric information in the problem situation from one form to the other. Skills in diagrammatic representation are not confined to converting text information into diagrammatic form. They are also required to generate goal-directed new information with the aid of general processing skills. For example, during the *planning* process, students need to identify key steps of proof development that could not be generated by algorithms. For example, in Figure 1, in order to prove that segments AY and BX are equal, the student had to plan to solve a sub-problem in finding the different segments that could be used to represent AY and BX. This process shows the role of planning in geometry proof building processes. The solution process related to the proof in Figure 1 also exemplifies the role of another GPS - *use of knowledge retrieval*. Retrieving appropriate knowledge and accurate use of those retrievals are in turn influenced by general skills. This process enabled students to access and retrieve the required theorems and geometric concepts, and to use them in generating new information in a goal-directed manner. In order for students to activate and use their prior knowledge in a goal-direct manner, we argue, they will have to call upon a second layer of skills which are *general or metacognitive* in nature.

We have indirect evidence here for the interplay between content knowledge and general processes as students attempt to develop proofs. The results h are consistent with findings of Hilbert, Renkl, Kessler and Reiss (2008) that studying examples of heuristics could help students' ability to solve geometry proof problems. Our results are also consistent with those of Yang (2012) in that metacognitive processes need to work in concert with cognitive processes in assisting students make sense of geometry proof problems.

How can we explain the role of the third predictor variable (Mathematical Reasoning Skills) in proof development? We have argued that domain-specific knowledge (GCK) as well as general processes (GPS) play a key role in directing and controlling the flow of given and new information within the problem space that is relevant to the construction of proof. However, what are the processes that aid students in the generation of new information from given information? It would seem that the use of reasoning skills plays a critical role here in this phase of the solution process. During the course of construction of proofs, students go through an iterative process in which relevant given information is used to drive the reasoning process and vice versa. The outcome of this process is the generation of new moves and sequential information production that feed off each other. Taken together, it would seem that the proving of a given statement or geometric relationship in a diagram can be expected to be dictated by the context-relevant content knowledge, general processes and reasoning..

An emerging issue here is domain-general or domain-specific nature of deductive reasoning that was activated during the construction of geometry proofs in the present study. Mathematical Reasoning (deductive reasoning in this case) might have both the attributes as students could be expected to activate such skills in the solution of non-geometric problems or, indeed, in solving geometric problems that do not involve proofs. Future studies could focus on the domain-specificity or otherwise of mathematical reasoning in the solution proof-type problems in general, and geometry in particular.

We set out to examine the relative impact of domain-specific and domain-general knowledge in the production of proofs for a series of geometry problems. The results of this study suggest that geometry proof development is a complex problem-solving activity that is underpinned by an ongoing interaction between *general processes* and *task-specific knowledge*. Of greater significance of this study is that we did pull out three core strands of knowledge that provided deeper insight into knowledge-related issue raised by Battista (2007). We suggest that further research is needed to untangle the relative role of these knowledge components and processes, and their interactions with a variety of geometry and other mathematical problems that involve the construction of proofs.

Our regression analysis of the four knowledge components could lead to the assumption that the development of formal geometry proofs follows a linear path. In adopting the regression model, we were driven to explore and highlight the relative contribution of content, general processes and reasoning on success in proof generation. In so doing, we do not underestimate the complexities underpinning proof development (Hanna, 2000) and do not claim that having these knowledge and skills is sufficient for students to become competent proof developers. These knowledge components interact and the paths of this interaction are far from linear.

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